DOMAIN-INDEPENDENT UPPER BOUNDS
FOR EIGENVALUES OF ELLIPTIC OPERATORS

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ABSTRACT. Let $\Omega \subseteq \mathbb{R}^m$ be a bounded open set, $\partial\Omega$ its boundary and Δ the Laplacian on $\mathbb{R}^m$. Consider the elliptic differential equation:

$$(1) \qquad -\Delta u = \lambda u \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega.$$

It is known that the eigenvalues, λ_i, of (1) satisfy

$$(2) \qquad \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \geq \frac{mn}{4}$$

provided that $\lambda_{n+1} > \lambda_n$.

In this paper we abstract the method used by Hile and Protter [2] to establish (2) and apply the method to a variety of second-order elliptic problems, in particular, to all constant coefficient problems. We then consider a variety of higher-order problems and establish an extension of (2) for problem (1) where the Laplacian is replaced by a more general operator in a Hilbert space.

1. INTRODUCTION

Let Δ be the Laplace operator in $\mathbb{R}^m$ and $\Omega \subseteq \mathbb{R}^m$ a bounded region and $\partial\Omega$ its boundary in which the eigenvalue problem

$$(1.1) \qquad -\Delta u = \lambda u \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega$$

has discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots$. Then the eigenvalues of (1.1) satisfy

$$(1.2) \qquad \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \geq \frac{mn}{4}.$$

The result (1.2) was established by Hile and Protter [2] and is a generalization of

$$(1.3) \qquad \lambda_{n+1} \leq \lambda_n + \frac{2}{n} \sum_{i=1}^{n} \lambda_i$$

for $m = 2$. Result (1.3) was established by Payne, Polya, and Weinberger [8]. Inequality (1.3) follows from (1.2) since the summand in (1.2) is bounded above

by $\lambda_i/(\lambda_{n+1} - \lambda_n)$ for each i. The interesting thing about (1.2) is that it leads to an upper bound on the $(n+1)$th eigenvalue of the Dirichlet problem, (1.1), in terms of the lower eigenvalues which is domain-independent. That is, there is no explicit dependence on the size or shape of the domain Ω in (1.2).

More recently Hile and Yeh [3] and Chen [1] considered polyharmonic operators of the form

$$(1.4) \qquad (-\Delta)^l u = \lambda u \quad \text{in } \Omega; \qquad u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{l-1} u}{\partial n^{l-1}} = 0 \quad \text{on } \partial\Omega,$$

where $\partial/\partial n$ denotes the outward-pointing normal derivative on $\partial\Omega$. Hile and Yeh [3] considered problem (1.4) with $l = 2$ and Chen [1] considered certain other even positive integers l. The result of [3] and [1] is that

$$(1.5) \qquad \left(\sum_{i=1}^{n} \frac{\lambda_i^{1/l}}{\lambda_{n+1} - \lambda_i} \right) \left(\sum_{i=1}^{l} \lambda_i \right)^{(l-1)/l} \geq \frac{m^2 n^{2l-1}}{4l(2l+m-2)}.$$

Again (1.5) has no explicit dependence on the domain Ω.

In this paper we abstract the method (Theorem 1) of [2] to a Hilbert space. After establishing the abstract generalization of this method in §2, we establish the result (1.2) as a strict inequality or one which is similar for a variety of second-order elliptic differential equations in one dependent variable. We establish (1.2) as a strict inequality for

$$(1.6) \qquad -e^{\mathbf{w} \cdot \mathbf{x}} \operatorname{div}(M^2 e^{-\mathbf{w} \cdot \mathbf{x}} \operatorname{grad} u) - \tfrac{1}{4}\|M\mathbf{w}\|^2 u = \lambda u \quad \text{in } \Omega;$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where M is a positive definite real symmetric matrix of constants, grad is the gradient, div is the divergence, $\mathbf{w}$ is a constant $\mathbf{w} \in \mathbb{R}^m$, $\mathbf{w} \cdot \mathbf{x}$ is the dot problem in $\mathbb{R}^m$ and $\|\cdot\|$ is the $\mathbb{R}^m$-Euclidean norm. Every (real) constant coefficient second-order elliptic problem with the boundary conditions of (1.6) can be put into the form (1.6), in particular those with first-order terms.

Results similar to (1.2) hold for

$$(1.7) \qquad -\Delta u - 2i\mathbf{w} \cdot \operatorname{grad} u = \lambda u \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega$$

in which $i^2 = -1$ (not to be confused with the index) and $\mathbf{w} \in \mathbb{R}^m$ is a constant. For problem (1.7), if the eigenvalues are discrete and $\lambda_{n+1} > \lambda_n$ then

$$(1.8) \qquad \sum_{i=1}^{n} \frac{\lambda_i - \|\mathbf{w}\|^2}{\lambda_{n+1} - \lambda_i} > \frac{mn}{4}$$

holds.

Finally the problem

$$(1.9) \qquad -\Delta u + \rho(x) u = \lambda u \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega$$

satisfies (1.2), where $\rho(x) \geq 0$ in Ω, and for the Sturm-Liouville problem

$$(1.10) \qquad -((\rho(x))^2 u')' + g(x) u = \lambda u \quad \text{in } I = (\alpha, \beta),$$

$$u(\alpha) = u(\beta) = 0,$$

where $\rho(x) > 0$ on I, the eigenvalues obey

$$(1.11) \qquad \sum_{i=1}^{n} \frac{\lambda_i - M}{\lambda_{n+1} - \lambda_i} \geq \frac{n}{4},$$

where M is any constant so that $M \leq g - \frac{1}{4}((\rho')^2 - (\rho^2)'')$ in I and the "prime" notation denotes the derivative.

The abstraction in Theorem 1 along with another theorem of this author [4] or [5] is used to establish an improvement of (1.5) for the problem

$$(1.12) \qquad p(-\Delta)u = \lambda u \quad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{l-1} u}{\partial n^{l-1}} = 0 \quad \text{on } \partial\Omega,$$

where p is a monic polynomial (leading coefficient is unity) of degree $l > 0$. In particular if p has nonnegative coefficients and $p(0) = 0$ then

$$(1.13) \qquad \left(\sum_{i=1}^{n} \frac{\lambda_i^{1/l}}{\lambda_{n+1} - \lambda_i} \right) \left(\sum_{i=1}^{n} \frac{m}{2} p'(\lambda_i^{1/l}) + \lambda_i^{1/l} p''(\lambda_i^{1/l}) \right) > \frac{m^2 n^2}{8}.$$

We also note that the result holds for polynomials in more general constant coefficient second-order equations and that the eigenvalues of some higher-order problems other than ones in the form of (1.12) satisfy similar inequalities.

Finally we consider the system of equations for an elastic medium

$$(1.14) \qquad -\Delta\mathbf{u} - \alpha \, \mathrm{grad}(\mathrm{div}(\mathbf{u})) = \Lambda\mathbf{u} \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega.$$

Equation (1.14) is a system of m partial differential equations. The symbol Δ is the operator which takes the Laplacian of each component of a vector field $\mathbf{u}$, and α is a constant. We establish that if $\Omega \subseteq \mathbb{R}^m$ is bounded and (1.14) has discrete eigenvalues $\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_n < \Lambda_{n+1} \leq \cdots$ and $\alpha \geq 0$ then

$$(1.15) \qquad \sum_{i=1}^{n} \frac{\Lambda_i}{\Lambda_{n+1} - \Lambda_i} \geq \frac{m^2 n}{4(m + \alpha)}$$

holds.

As noted before, none of the results of this paper have any explicit dependence on the domain, Ω, except for the dimension m. We begin §2 with the abstraction of the method of [2].

2. AN ABSTRACT THEOREM

Our results depend on a rather complicated-looking theorem. It is given abstractly with applications in subsequent sections. For linear operators L and M we define the *commutator*, $[L, M]$, by $[L, M]u = LMu - MLu$. We have the following theorem.

Theorem 1. *Let $A: D \subseteq \mathscr{H} \to \mathscr{H}$ be a selfadjoint operator in a real or complex Hilbert space $\mathscr{H}$, with inner product $\langle \cdot, \cdot \rangle$, which is semibounded below with*

discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots$. *Let* $\{T_j : D \to \mathcal{H}\}_{j=1}^M$ *be any collection of skew-symmetric operators and* $\{B_j : T_j(D) \to \mathcal{H}\}_{j=1}^M$ *any collection of symmetric operators which leave* D *invariant. Denote the normalized eigenvectors of* A *by* $\{u_i\}_{i=1}^\infty$, *where* u_i *corresponds to* λ_i. *Suppose that* $\lambda_{n+1} > \lambda_n$. *If for some* i *and* j, $T_j u_i \neq 0$ *and if there is a* $\gamma > 0$ *and* $\beta > \lambda_n$ *so that*

$$(2.1) \qquad \gamma \sum_{i=1}^n \sum_{j=1}^M \langle [T_j, B_j] u_i, u_i \rangle + \gamma^2 \sum_{i=1}^n \sum_{j=1}^M \frac{1}{\beta - \lambda_i} \langle T_j u_i, T_j u_i \rangle$$

$$= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^M \langle [[A, B_j], B_j] u_i, u_i \rangle$$

then $\beta \geq \lambda_{n+1}$. *Moreover if* $T_j u_i \notin D$ *for all* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, M$ *we have* $\beta > \lambda_{n+1}$.

Proof. We construct a collection of test functions, ϕ_{ij}, which are orthogonal to each of the first n eigenvectors and apply well-known minimizing properties of λ_{n+1}. For each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, M$, define

$$(2.2) \qquad \phi_{ij} = B_j u_i - \sum_{k=1}^n a_{jik} u_k,$$

where

$$(2.3) \qquad a_{jik} = \langle B_j u_i, u_k \rangle.$$

From (2.2) and (2.3) we see that

$$(2.4) \qquad \langle \phi_{ij}, u_k \rangle = 0 \quad \text{for all } i, k = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, M.$$

Then for each i, j we have

$$(2.5) \qquad \lambda_{n+1} \langle \phi_{ij}, \phi_{ij} \rangle \leq \langle A \phi_{ij}, \phi_{ij} \rangle$$

since each ϕ_{ij} is orthogonal to each of the first n eigenvectors of A by (2.4).

Next we work on the right-hand side of (2.5). By the definition of the ϕ_{ij} and (2.4) we have

$$(2.6) \qquad \langle A \phi_{ij}, \phi_{ij} \rangle = \langle A B_j u_i, \phi_{ij} \rangle.$$

We apply the definition of the commutator and the fact that $A u_i = \lambda_i u_i$ to the right-hand side of (2.6) we obtain

$$(2.7) \qquad \langle A \phi_{ij}, \phi_{ij} \rangle = \langle B_j A u_i, \phi_{ij} \rangle + \langle [A, B_j] u_i, \phi_{ij} \rangle$$

$$= \lambda_i \langle B_j u_i, \phi_{ij} \rangle + \langle [A, B_j] u_i, \phi_{ij} \rangle.$$

Since $\langle B_j u_i, \phi_{ij} \rangle = \langle \phi_{ij}, \phi_{ij} \rangle$ by definition of ϕ_{ij} and the orthogonality condition, (2.4), we can rewrite (2.7) as

$$(2.8) \qquad \langle A \phi_{ij}, \phi_{ij} \rangle = \lambda_i \langle \phi_{ij}, \phi_{ij} \rangle + \langle [A, B_j] u_i, \phi_{ij} \rangle.$$

Substitution of the definition of ϕ_{ij} into the rightmost term of (2.8), substitution of the result into the right-hand side of (2.5) and summing this over all i yields

$$(2.9) \qquad \lambda_{n+1} \sum_{i=1}^{n} \langle \phi_{ij}, \phi_{ij} \rangle \leq \sum_{i=1}^{n} \lambda_i \langle \phi_{ij}, \phi_{ij} \rangle + \sum_{i=1}^{n} \langle [A, B_j]u_i, B_j u_i \rangle$$

$$- \sum_{i=1}^{n} \sum_{k=1}^{n} \langle [A, B_j]u_i, a_{jik}u_k \rangle .$$

The right-hand side of (2.9) is real so (2.9) still holds if we take the real part of the right-hand side. (Notice that in a real Hilbert space the operation of obtaining the real part still makes sense!) As it is, the real part of the double sum is zero since

$$(2.10) \ \operatorname{Re} \sum_{k,i=1}^{n} \langle [A, B_j]u_i, a_{jik}u_k \rangle$$

$$= \frac{1}{2} \left[\sum_{i,k=1}^{n} \langle [A, B_j]u_i, a_{jik}u_k \rangle + \overline{\sum_{i,k=1}^{n} \langle [A, B_j]u_i, a_{jik}u_k \rangle} \right]$$

$$= \frac{1}{2} \left[\sum_{i,k=1}^{n} \langle [A, B_j]u_i, a_{jik}u_k \rangle + \sum_{i,k=1}^{n} \langle a_{jik}u_k, [A, B_j]u_i \rangle \right]$$

$$= \frac{1}{2} \left[\sum_{i,k=1}^{n} \langle [A, B_j]u_i, a_{jik}u_k \rangle + \sum_{i,k=1}^{n} \langle -[A, B_j]u_k, a_{jki}u_i \rangle \right] .$$

The last step is justified since the adjoint $[A, B_j]^* = -[A, B_j]$ whenever A and B_j are symmetric and the fact that $\overline{a_{jik}} = a_{jki}$ for all i, k, j. Switching i and k in either double sum in the right-hand side of (2.10) establishes that the real part of the double sum is (2.9) is, indeed, zero.

After taking real parts in (2.9) we have

(2.11)
$$\lambda_{n+1} \sum_{i=1}^{n} \langle \phi_{ij}, \phi_{ij} \rangle \leq \sum_{i=1}^{n} \lambda_i \langle \phi_{ij}, \phi_{ij} \rangle + \frac{1}{2} \sum_{i=1}^{n} (\langle [A, B_j]u_i, B_j u_i \rangle + \langle B_j u_i, [A, B_j]u_i \rangle).$$

The symmetry of B_j, skew symmetry of $[A, B_j]$ and the definition of the commutator are used to establish that

(2.12)
$$\frac{1}{2} \sum_{i=1}^{n} (\langle [A, B_j]u_i, B_j u_i \rangle + \langle B_j u_i, [A, B_j]u_i \rangle) = -\frac{1}{2} \sum_{i=1}^{n} \langle [[A, B_j], B_j]u_i, u_i \rangle .$$

Thus (2.11) becomes

$$(2.13) \qquad \lambda_{n+1} \sum_{i=1}^{n} \langle \phi_{ij}, \phi_{ij} \rangle \leq \sum_{i=1}^{n} \lambda_i \langle \phi_{ij}, \phi_{ij} \rangle - \frac{1}{2} \sum_{i=1}^{n} \langle [[A, B_j], B_j]u_i, u_i \rangle .$$

Next let $\beta > \lambda_n$ be a real constant and subtract $\beta \sum_{i=1}^{n} \langle \phi_{ij}, \phi_{ij} \rangle$ from both sides of (2.13) and sum the result over all j. This yields

$$(2.14) \quad (\lambda_{n+1} - \beta) \sum_{i=1}^{n} \sum_{j=1}^{M} \langle \phi_{ij}, \phi_{ij} \rangle$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{M} (\lambda_i - \beta) \langle \phi_{ij}, \phi_{ij} \rangle - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{M} \langle [[A, B_j], B_j] u_i, u_i \rangle.$$

At this point we want to find an upper bound for $\sum_{i=1}^{n} \sum_{j=1}^{M} (\lambda_i - \beta) \langle \phi_{ij}, \phi_{ij} \rangle$ expressed in terms of the u_i and certain operators. To do this, recall that in any Hilbert space if $\delta > 0$ we have

$$(2.15) \qquad\qquad \mathrm{Re} \langle u, v \rangle \leq \frac{\delta}{2} \langle u, u \rangle + \frac{1}{2\delta} \langle v, v \rangle$$

with strict inequality except when $\delta u = v$. In particular, if u and v are linearly independent, inequality (2.15) is strict. For each j we have

$$(2.16)$$

$$\sum_{i=1}^{n} \mathrm{Re} \langle \phi_{ij}, T_j u_i \rangle = \frac{1}{2} \sum_{i=1}^{n} (\langle B_j u_i, T_j u_i \rangle + \langle T_j u_i, B_j u_i \rangle)$$

$$- \sum_{i,k=1}^{n} (\langle a_{jik} u_k, T_j u_i \rangle + \langle T_j u_i, a_{jik} u_k \rangle)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \langle [B_j, T_j] u_i, u_i \rangle - \mathrm{Re} \sum_{i=1}^{n} \sum_{k=1}^{n} \langle a_{jik} u_k, T_j u_i \rangle.$$

Since $a_{jik} = \overline{a_{jki}}$ for each j, k and i and T_j is skew-symmetric, the double sum in (2.16) must be zero as it simply represents the sum of all the entries of a skew-symmetric matrix. Thus we have

$$(2.17) \qquad\qquad \mathrm{Re} \sum_{i=1}^{n} \langle \phi_{ij}, T_j u_i \rangle = \frac{1}{2} \sum_{i=1}^{n} \langle [B_j, T_j] u_i, u_i \rangle.$$

Application of (2.15) to the left-hand side of (2.17) with $\delta = (\beta - \lambda_i)/\gamma$ yields

$$(2.18)$$

$$\frac{1}{2} \sum_{i=1}^{n} \langle [B_j, T_j] u_i, u_i \rangle \leq \frac{1}{2\gamma} \sum_{i=1}^{n} (\beta - \lambda_i) \langle \phi_{ij}, \phi_{ij} \rangle + \frac{\gamma}{2} \sum_{i=1}^{n} \frac{1}{\beta - \lambda_i} \langle T_j u_i, T_j u_i \rangle.$$

Inequality (2.18) holds for all positive γ. Rearrangement of (2.18) and summing over j yields the desired upper bound:

$$(2.19) \qquad \sum_{j=1}^{M} \sum_{i=1}^{n} (\lambda_i - \beta) \langle \phi_{ij}, \phi_{ij} \rangle$$

$$\leq -\gamma \sum_{j=1}^{M} \sum_{i=1}^{n} \langle [B_j, T_j] u_i, u_i \rangle + \gamma^2 \sum_{j=1}^{M} \sum_{i=1}^{n} \frac{1}{\beta - \lambda_i} \langle T_j u_i, T_j u_i \rangle.$$

Substitution of (2.19) into (2.14) yields

$$(2.20) \quad (\lambda_{n+1} - \beta) \sum_{i=1}^{n} \sum_{j=1}^{M} \langle \phi_{ij}, \phi_{ij} \rangle \leq - \gamma \sum_{j=1}^{M} \sum_{i=1}^{n} \langle [B_j, T_j] u_i, u_i \rangle$$

$$+ \gamma^2 \sum_{j=1}^{M} \sum_{i=1}^{n} \frac{1}{\beta - \lambda_i} \langle T_j u_i, T_j u_i \rangle$$

$$- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{M} \langle [[A, B_j], B_j] u_i, u_i \rangle.$$

The right-hand side of (2.20) depends on β since we have assumed that $T_j u_i \neq 0$ for some i and j. So if we can choose $\beta > \lambda_n$ and $\gamma > 0$ so that the right-hand side of (2.20) is zero then a larger β makes the right-hand side negative, holding γ constant. In particular the left-hand side of (2.20) can be made negative, so the quantity $\sum_{j=1}^{M} \sum_{i=1}^{n} \langle \phi_{ij}, \phi_{ij} \rangle > 0$. Thus when the right-hand side of (2.20) is equal to zero we must have $\lambda_{n+1} - \beta \leq 0$. In other words, $\beta \geq \lambda_{n+1}$. Now the right-hand side of (2.20) is equal to zero precisely when

$$(2.21) \qquad \gamma \sum_{j=1}^{M} \sum_{i=1}^{n} \langle [T_j, B_j] u_i, u_i \rangle + \gamma^2 \sum_{i=1}^{n} \frac{1}{\beta - \lambda_i} \sum_{j=1}^{M} \langle T_j u_i, T_j u_i \rangle$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{M} \langle [[A, B_j], B_j] u_i, u_i \rangle.$$

Equalities (2.21) and (2.1) are the same, with the above discussion establishing the weak inequality.

Notice that the upper bound, (2.19), comes from (2.15) and (2.16). In particular if ϕ_{ij} and $T_j u_i$ are linearly independent for some i and j the upper bound, (2.19), is actually a strict inequality. We have argued above that for some i and j, $\phi_{ij} \neq 0$. Under the additional assumption that $T_j u_i \notin D$ for any i and j, ϕ_{ij} and $T_j u_i$ cannot be linearly dependent since $\phi_{ij} \in D$. Thus if $T_j u_i \notin D$ for all $i = 1, 2, \ldots, n$ and $j = 1, \ldots, M$ we must have $\beta > \lambda_{n+1}$ whenever β satisfies (2.1). $\square$

Theorem 1 is an abstraction of a method used first by Payne, Polya, and Weinberger [8], refined later by Hile and Protter [2] and effectively used again in [3], [1], and [4]. The result does not look too useful since we do not yet know how to choose the T_j and B_j. Clearly though, we need to be able to choose the B_j and T_j so that the inner products can be calculated. As it is, we can often choose the T_j and B_j so that $[T_j, B_j] u = -u$, $\sum_{j=1}^{M} \langle T_j u, T_j u \rangle = \langle Au, u \rangle$ and $[[A, B_j], B_j] u = -2u$ as follows.

Proposition 1. *If A, T_j and B_j are as in Theorem 1, $A = -\sum_{j=1}^{M} T_j^2$ and if $[T_j, B_i] u = -\delta_{ij} u$ where δ_{ij} is the Kronecker symbol then*

$$(2.22) \qquad\qquad [[A, B_j], B_j] u = -2u.$$

Proof.

$$[A, B_j]u = -\sum_{k=1}^{M}[T_k^2, B_j]u = -\sum_{k=1}^{M}(T_k[T_k, B_j] + [T_k, B_j]T_k)u$$

$$= -T_j[T_j, B_j]u - [T_j, B_j]T_ju = 2T_ju.$$

The sum collapses since $[T_k, B_j] = 0$ if $k \neq j$. Thus $[[A, B_j], B_j]u = [2T_j, B_j]u = 2[T_j, B_j]u = -2u$. $\square$

Although it is not required, in our applications, A is a differential operator, the T_j are differential operators and the B_j are multiplication operators. Proposition 1 effectively assures us that if $A = -\sum_{j=1}^{M} T_j^2$ where the T_j are skew-symmetric and if a certain system of differential equations ($[T_j, B_i]u = -\delta_{ij}u$) can be solved then we can expect to obtain a useful inequality from (2.1).

3. SECOND-ORDER DIFFERENTIAL EQUATIONS

We begin by considering the most general real constant coefficient elliptic problem with Dirichlet boundary conditions:

$$(3.1) \qquad Au = -e^{\mathbf{w} \cdot \mathbf{x}} \operatorname{div}(M^2 e^{-\mathbf{w} \cdot \mathbf{x}} \operatorname{grad} u) - \tfrac{1}{4}\|M\mathbf{w}\|^2 u = \lambda u \quad \text{in } \Omega;$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^m$ is bounded, M is a constant $m \times m$ (real) symmetric positive definite matrix and $\mathbf{w} \in \mathbb{R}^m$ is a constant vector. We apply Theorem 1 to problem (3.1) using the following lemmas.

Lemma 1. *Let D be a positive diagonal matrix and U a real orthogonal matrix so that $M = U^{-1}DU$. For $j = 1, 2, \ldots, m$, the operators*

$$(3.2) \qquad T_ju = \tfrac{1}{2}\mathbf{e}_j \cdot (e^{\mathbf{w} \cdot \mathbf{x}}(DU \operatorname{grad}(e^{-\mathbf{w} \cdot \mathbf{x}}u)) + DU \operatorname{grad}(u))$$

are skew-symmetric under the boundary conditions of (3.1) and inner product

$$(3.3) \qquad \langle u, v \rangle = \int_{\Omega} uve^{-\mathbf{w} \cdot \mathbf{x}}.$$

Moreover we have

$$(3.4) \qquad \sum_{j=1}^{m} \langle T_ju, T_ju \rangle = \langle Au, u \rangle$$

and if $Au = \lambda u$ and $u \neq 0$ then T_ju is not identically zero on $\partial\Omega$.

Proof. We need only show that $\langle T_ju, v \rangle = -\langle u, T_jv \rangle$, where u and v satisfy

the boundary conditions of (3.1). But

$$(3.5) \quad \langle T_j u, v \rangle = \frac{1}{2} \int_\Omega \mathbf{e}_j \cdot (DU \operatorname{grad}(e^{-\mathbf{w}\cdot\mathbf{x}} u)v + DU \operatorname{grad}(u)v e^{-\mathbf{w}\cdot\mathbf{x}})$$

$$= -\frac{1}{2} \int_\Omega \mathbf{e}_j \cdot (DUe^{-\mathbf{w}\cdot\mathbf{x}} u \operatorname{grad}(v) + DU u \operatorname{grad}(e^{-\mathbf{w}\cdot\mathbf{x}} v))$$

$$= -\frac{1}{2} \int_\Omega \mathbf{e}_j \cdot u(e^{\mathbf{w}\cdot\mathbf{x}}(DU \operatorname{grad}(e^{-\mathbf{w}\cdot\mathbf{x}} v)) + DU \operatorname{grad} v)e^{-\mathbf{w}\cdot\mathbf{x}}$$

$$= -\langle u, T_j v \rangle.$$

To establish (3.4), notice that (3.2) can be simplified to

$$(3.6) \qquad\qquad T_j u = \frac{1}{2} \mathbf{e}_j \cdot (2DU \operatorname{grad} u - DU\mathbf{w}u).$$

Thus

$$(3.7) \qquad \sum_{j=1}^m \langle T_j u, T_j u \rangle = \frac{1}{4} \int_\Omega \|2DU \operatorname{grad} u - DU\mathbf{w}u\|^2 e^{-\mathbf{w}\cdot\mathbf{x}}.$$

Using the norm-preserving property of U and multiplying out the norm in (3.7) we see that

$$\tfrac{1}{4}\|2DU \operatorname{grad} u - DU\mathbf{w}u\|^2 = \|DU \operatorname{grad} u\|^2 - DU \operatorname{grad} u \cdot DU\mathbf{w}u + \tfrac{1}{4}\|DU\mathbf{w}\|^2 u^2$$

$$= \|M \operatorname{grad} u\|^2 - M \operatorname{grad} u \cdot M\mathbf{w}u + \tfrac{1}{4}\|M\mathbf{w}\|^2 u^2.$$

Thus (3.7) becomes

$$(3.8) \quad \sum_{j=1}^m \langle T_j u, T_j u \rangle = \int_\Omega \|M \operatorname{grad} u\|^2 e^{-\mathbf{w}\cdot\mathbf{x}} - \int_\Omega M \operatorname{grad} u \cdot M\mathbf{w}u e^{-\mathbf{w}\cdot\mathbf{x}}$$

$$+ \frac{1}{4}\|M\mathbf{w}\|^2 \int_\Omega u^2 e^{-\mathbf{w}\cdot\mathbf{x}}.$$

Notice that by the divergence theorem and boundary conditions we have

$$(3.9) \qquad 0 = \int_\Omega \operatorname{div}(uM^2 e^{-\mathbf{w}\cdot\mathbf{x}} \operatorname{grad} u)$$

$$= \int_\Omega \operatorname{grad} u \cdot M^2 \operatorname{grad}(u)e^{-\mathbf{w}\cdot\mathbf{x}} + \int_\Omega u \operatorname{div}(M^2 e^{-\mathbf{w}\cdot\mathbf{x}} \operatorname{grad} u)$$

$$= \int_\Omega \|M \operatorname{grad} u\|^2 e^{-\mathbf{w}\cdot\mathbf{x}} + \int_\Omega \operatorname{div}(M^2 e^{-\mathbf{w}\cdot\mathbf{x}} \operatorname{grad} u)u.$$

Thus

$$\int_\Omega \|M \operatorname{grad} u\|^2 e^{-\mathbf{w}\cdot\mathbf{x}} = -\int_\Omega \operatorname{div}(M^2 e^{-\mathbf{w}\cdot\mathbf{x}} \operatorname{grad} u)u.$$

Also notice that when u satisfies the boundary conditions of (3.1)

$$(3.10) \qquad 0 = \int_\Omega \frac{1}{2} M^2 \mathbf{w} \cdot \operatorname{grad}(u^2 e^{-\mathbf{w}\cdot\mathbf{x}})$$

$$= \int_\Omega \frac{1}{2} M^2 \mathbf{w} \cdot \operatorname{grad}(u)2u e^{-\mathbf{w}\cdot\mathbf{x}} - \frac{1}{2} \int_\Omega M^2 \mathbf{w} \cdot \mathbf{w}u^2 e^{-\mathbf{w}\cdot\mathbf{x}}.$$

Upon rearranging (3.10) using the symmetry of M we obtain

(3.11) $\int_\Omega M\,\text{grad}(u)\cdot M\mathbf{w}ue^{-\mathbf{w}\cdot\mathbf{x}} = \int_\Omega M^2\mathbf{w}\cdot\text{grad}(u)(ue^{-\mathbf{w}\cdot\mathbf{x}})$

$$= \frac{1}{2}\|M\mathbf{w}\|^2\int_\Omega u^2 e^{-\mathbf{w}\cdot\mathbf{x}}\,.$$

Placing the results of (3.9) and (3.11) into (3.8) shows that (3.4) holds.

Finally $T_j u$ is a linear combination of u and a directional derivative of u. Thus the boundary conditions are such that $T_j u$ will satisfy them only if $\text{grad}\,u = 0$ on $\partial\Omega$. This is impossible for an eigenvector of A. $\square$

Lemma 2. *There are real-valued functions* ϕ_i, $i = 1, 2, \ldots, m$, *so that* $T_j(\phi_i u)$ $- \phi_i T_j u = -\delta_{ij}u$ *if* u *satisfies the boundary conditions of* (3.1). *Define the functions* B_j *of Theorem* 1 *by*

(3.12) $B_j u = \phi_j u \quad \text{for } j = 1, 2, \ldots, m.$

The operators B_j *are symmetric in* $L^2(\Omega)$ *and leave the boundary conditions of* (3.1) *invariant. Moreover*

(3.13) $[T_j, B_i]u = -\delta_{ij}u.$

Proof. Using the formulation of T_j in (3.6) we see that

(3.14) $T_j(\phi u) - \phi T_j u = \mathbf{e}_j\cdot DUu\,\text{grad}\,\phi$

for any differentiable function ϕ. We thus need only show that there are functions ϕ_i so that

(3.15) $\mathbf{e}_j\cdot DU\,\text{grad}\,\phi_i = -\delta_{ij}\,.$

But (3.15) is equivalent to

(3.16) $U^{-1}D\mathbf{e}_j\cdot\text{grad}\,\phi_i = -\delta_{ij}\,.$

Thus the ϕ_i need only have constant gradient in a fixed direction since $U^{-1}D$ is a constant matrix. Such ϕ_i always exist. The remainder of the lemma follows easily. $\square$

We have enough now to prove

Theorem 2. *If problem* (3.1) *has discrete spectrum* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n < \lambda_{n+1} < \cdots$ *and is selfadjoint with inner product* (3.3) *then the eigenvalues of* (3.1) *satisfy*

(3.17) $\displaystyle\sum_{i=1}^n \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} > \frac{mn}{4}\,.$

Proof. We take the T_j and B_j in Theorem 1 to be those of Lemmas 1 and 2. Evidently these T_j and B_j satisfy Proposition 1. With these T_j and B_j, (2.1) becomes

(3.18) $-nm\gamma + \gamma^2\displaystyle\sum_{i=1}^n \frac{\lambda_i}{\beta - \lambda_i} = -nm$

This follows because

$$\langle [T_j, B_j] u_i, u_i \rangle = -1, \qquad \sum_{j=1}^{m} \langle T_j u_i, T_j u_i \rangle = \langle A u_i, u_i \rangle = \lambda_i,$$

and

$$\tfrac{1}{2} \langle [[A, B_j], B_j] u_i, u_i \rangle = -1$$

which follow from Lemma 2, Lemma 1, and Proposition 1 respectively. The left-hand side of (3.18) is a function of γ. In particular if we minimize the left-hand side of (3.18) in γ, (3.18) becomes

$$(3.19) \qquad -\frac{m^2 n^2}{4 \sum_{i=1}^{n} \lambda_i/(\beta - \lambda_i)} = -mn,$$

the left-hand side of (3.19) being the minimum value of the left-hand side of (3.18) with

$$\gamma = \frac{mn}{2 \sum_{i=1}^{n} \lambda_i/(\beta - \lambda_i)}.$$

Rearrangement of (3.19) yields

$$(3.20) \qquad \sum_{i=1}^{n} \frac{\lambda_i}{\beta - \lambda_i} = \frac{mn}{4}.$$

There is always a (unique) solution $\beta > \lambda_n$ for (3.20). Moreover such a $\beta > \lambda_{n+1}$ by Theorem 1 and Lemma 1. Thus $\beta - \lambda_i > \lambda_{n+1} - \lambda_i$ and so

$$(3.21) \qquad \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} > \frac{mn}{4}. \qquad \square$$

Remark. Equation (3.1) includes all real constant coefficient second-order elliptic problems with $u = 0$ on $\partial\Omega$. Writing out (3.1) yields

$$(3.22) \qquad Au = -\operatorname{div}(M^2 \operatorname{grad} u) + M^2 \mathbf{w} \cdot \operatorname{grad} u - \tfrac{1}{4} \|M\mathbf{w}\|^2 u = \lambda u.$$

Every real constant coefficient second-order elliptic operator can be written as

$$(3.23) \qquad Eu = -\operatorname{div}(M^2 \operatorname{grad} u) + \mathbf{v} \cdot \operatorname{grad} u + cu = \lambda u$$

for constants $\mathbf{v} \in \mathbb{R}^m$ and $c \in \mathbb{R}$. Since M^2 is invertible we can always obtain $\mathbf{w}$ from $\mathbf{v}$ by $\mathbf{w} = (M^2)^{-1}\mathbf{v}$. Moreover, part of the constant c may be incorporated into the eigenvalue λ. Thus we can rewrite (3.23) as

$$(3.24) \qquad \widetilde{E}u = -\operatorname{div}(M^2 \operatorname{grad} u) + \mathbf{v} \cdot \operatorname{grad} u - \tfrac{1}{4}\|M^{-1}\mathbf{v}\|^2 u$$
$$= (\lambda - c + \tfrac{1}{4}\|M^{-1}\mathbf{v}\|^2)u = \mu u.$$

Now $\widetilde{E}$ is in the form of (3.22) and its eigenvalues differ from those of (3.23) by the constant $-c + \tfrac{1}{4}\|M^{-1}\mathbf{v}\|^2$. Thus, as an easy corollary to Theorem 2, the eigenvalues of (3.22) satisfy

$$(3.25) \qquad \sum_{i=1}^{n} \frac{\lambda_i - c + \tfrac{1}{4}\|M^{-1}\mathbf{v}\|^2}{\lambda_{n+1} - \lambda_i} > \frac{mn}{4}.$$

The result of Hile and Protter [2] is also an easy corollary of Theorem 2.

Corollary 1. *If the eigenvalues of*

(3.26) $\qquad\qquad -\Delta u = \lambda u \quad in\ \Omega \subseteq \mathbb{R}^m;\qquad u = 0 \quad on\ \partial\Omega$

are discrete $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots$ *then*

(3.27) $$\sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} > \frac{mn}{4}.$$

Inequality (3.27) was only given as the weak inequality

$$\sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \geq \frac{mn}{4}$$

in [2] and, thus, has been strengthened here. Inequality (3.27) easily follows from Theorem 2 taking M to be the identity matrix and $\mathbf{w} = \mathbf{0}$.

It is also possible to take the vector $\mathbf{w}$ in Theorem 2 to be complex, but then inner product (3.3) must be changed and the form of (3.1) must be changed somewhat. The following shows what to do in an easy problem with first-order terms with purely imaginary coefficients.

Theorem 3. *Let* $\Omega \subseteq \mathbb{R}^m$ *be bounded. Let* $\mathbf{w} \in \mathbb{R}^m$ *be a constant vector. Consider the eigenvalue problem*

(3.28) $\qquad Au = -\Delta u - 2i\mathbf{w}\cdot\operatorname{grad} u = \lambda u \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega.$

The operator A *is symmetric in the inner product*

(3.29) $$\langle u, v \rangle = \int_{\Omega} u\bar{v}$$

subject to the boundary conditions of (3.28)*. If the eigenvalues of* (3.28) *are discrete,* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots$, *then the following inequality is satisfied:*

(3.30) $$\sum_{i=1}^{n} \frac{\lambda_i - \|\mathbf{w}\|^2}{\lambda_{n+1} - \lambda_i} > \frac{mn}{4}.$$

Proof. The symmetry of A is easy. Again we use Theorem 1 with

(3.31a) $$T_j u = \frac{\partial u}{\partial x_j} + i w_j u$$

and

(3.31b) $\qquad\qquad B_j u = -x_j u \quad for\ j = 1, 2, \ldots, m,$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_m)$. It is easy to see that the T_j are all skew-symmetric and the B_j are symmetric and preserve the boundary conditions. As before, we have $[T_j, B_i]u = -\delta_{ij}u$ and

(3.32) $$\sum_{j=1}^{m} \langle T_j u, T_j u \rangle = \sum_{j=1}^{m} \int_{\Omega} \left(\frac{\partial u}{\partial x_j} + i w_j u\right) \overline{\left(\frac{\partial u}{\partial x_j} + i w_j u\right)}$$

$$= \int_{\Omega} (-\Delta u - 2i\mathbf{w}\cdot\operatorname{grad} u + \|\mathbf{w}\|^2 u)\bar{u}$$

$$= \langle Au, u \rangle + \|\mathbf{w}\|^2 \langle u, u \rangle.$$

With the foregoing and Proposition 1, equation (2.1) becomes (taking γ as before)

$$(3.33) \qquad \sum_{i=1}^{n} \frac{\lambda_i + \|\mathbf{w}\|^2}{\lambda_{n+1} - \lambda_i} > \sum_{i=1}^{n} \frac{\lambda_i - \|\mathbf{w}\|^2}{\beta - \lambda_i} = \frac{mn}{4}. \quad \square$$

If an elliptic operator has any variable coefficients, a completely general solution is not easy. Several types of nonconstant coefficient problems are considered in the remainder of this section.

Theorem 4. *Let $\Omega \subseteq \mathbb{R}^m$ be bounded and consider the eigenvalue problem*

$$(3.34) \qquad Au = -\Delta u + \rho(x)u = \lambda u \quad in \ \Omega; \qquad u = 0 \quad on \ \partial\Omega$$

where ρ is a nonnegative real-valued function on Ω. If Ω is such that (3.34) is selfadjoint in inner product (3.29) and its eigenvalues, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n < \lambda_{n+1} \le \cdots$, are discrete then the inequality

$$(3.35) \qquad \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \ge \frac{nm}{4}$$

holds.

Proof. The operator A is symmetric in the inner product of (3.29). For this problem we may choose

$$(3.36a) \qquad T_j u = \partial u / \partial x_j \quad for \ j = 1, 2, \ldots, m,$$

$$(3.36b) \qquad T_{m+1} u = \left(i\sqrt{\rho(x)} \right) u.$$

Also we take

$$(3.37a) \qquad B_j u = -x_j u \quad for \ j = 1, 2, \ldots, m,$$

$$(3.37b) \qquad B_{m+1} u = u.$$

It is easily seen that the T_j are skew-symmetric and the B_j are symmetric with the given boundary conditions. We also have

$$(3.38) \qquad \sum_{j=1}^{m+1} \langle T_j u, T_j u \rangle = \langle Au, u \rangle.$$

The commutators $[T_j, B_j]u = -u$ for $j = 1, 2, \ldots, m$ and $[T_{m+1}, B_{m+1}] = 0$, $[A, B_j] = 2T_j$ for $j = 1, 2, \ldots, m$ and $[A, B_{m+1}] = 0$. With the above commutators, (2.1) becomes

$$(3.39) \qquad \gamma(-nm) + \gamma^2 \sum_{i=1}^{n} \frac{\lambda_i}{\beta - \lambda_i} = -nm.$$

Since (3.39) matches (3.18) the remainder of the proof follows that of Theorem 2. $\square$

Theorem 4 may be extended to arbitrary functions ρ as follows:

Corollary 2. *Suppose that the hypotheses of Theorem* 4 *hold except that we only have* $\rho(x) \geq M$ *for all* $x \in \Omega$. (*The constant* M *may be either positive or negative.*) *Then the eigenvalues of* (3.34) *satisfy*

$$(3.40) \qquad \sum_{i=1}^{n} \frac{\lambda_i - M}{\lambda_{n+1} - \lambda_i} \geq \frac{mn}{4} .$$

Proof. We have $\rho(x) - M \geq 0$ and thus

$$(3.41) \qquad \tilde{A}u = -\Delta u + (\rho(x) - M)u = (\lambda - M)u = \mu u$$

is in the form of (3.34) and has eigenvalues $\mu_i = \lambda_i - M$. Since the eigenvalues μ_i satisfy (3.35), replacing μ_i with $\lambda_i - M$ yields the result (3.40). □

Remark. Corollary 2 improves (3.35) even for $\rho(x) \geq 0$ provided that $\rho(x) \geq M > 0$ in Ω.

In general it is difficult at best to establish results similar to the above for elliptic problems with variable coefficients which are differentiated. However in the case of an ordinary differential equation the solution is easy.

Theorem 5. *Let* $I = (\alpha, \beta) \subseteq \mathbb{R}$ *be an open interval. Let* $\rho(x) > 0$ *in* I, *and let* $g(x)$ *be any real-valued function on* I. *If the eigenvalues of the Sturm-Liouville problem*

$$(3.42) \qquad Au = -((\rho(x))^2 u')' + g(x)u = \lambda u \quad \text{in } I, \qquad u(\alpha) = u(\beta) = 0$$

are discrete $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots$ *and if* $g - \frac{1}{4}((\rho')^2 - (\rho^2)'') \geq M$, *a constant, on* I *then the eigenvalues satisfy*

$$(3.43) \qquad \sum_{i=1}^{n} \frac{\lambda_i - M}{\lambda_{n+1} - \lambda_i} \geq \frac{n}{4} .$$

Proof. The operator A is symmetric in the inner product $\langle u, v \rangle = \int_I u\bar{v}$. Here we choose

$$(3.44\text{a}) \qquad T_1 u = \tfrac{1}{2}(\rho(x)u' + (\rho(x)u)') \quad \text{and} \quad B_1 u = \phi u,$$

$$(3.44\text{b}) \qquad T_2 u = \left(i\sqrt{g - \tfrac{1}{4}((\rho')^2 - (\rho^2)'')} - M \right) u \quad \text{and} \quad B_2 u = u,$$

where $\rho\phi' = -1$. The operators T_j are easily shown to be skew-symmetric and B_j are clearly symmetric. We also have

$$(3.45\text{a}) \qquad \langle T_1 u, T_1 u \rangle = \int_I |\rho u' + \tfrac{1}{2}\rho' u|^2$$

$$= \int_I -(\rho^2 u')'\bar{u} + \frac{1}{4}\int_I ((\rho')^2 - (\rho^2)'')|u|^2$$

and

$$(3.45\text{b}) \qquad \langle T_2 u, T_2 u \rangle = \int_I (g - \tfrac{1}{4}((\rho')^2 - (\rho^2)'') - M)|u|^2 .$$

Thus by equations (3.45) we see that

(3.46) $$\langle T_1 u, T_1 u\rangle + \langle T_2 u, T_2 u\rangle = \langle Au, u\rangle - M\langle u, u\rangle.$$

It is easy to see that

(3.47a) $$[T_1, B_1]u = -u,$$

(3.47b) $$[T_2, B_2]u = 0,$$

(3.47c) $$[A, B_1]u = 2T_1 u,$$

(3.47d) $$[A, B_2]u = 0.$$

Combining equations (3.47) and putting them into (2.1) we get

(3.48) $$-\gamma n + \gamma^2 \sum_{i=1}^{n} \frac{\lambda_i - M}{\lambda_{n+1} - \lambda_i} = -n.$$

This is in the same form as (3.18) with $m = 1$. Again the remainder follows the proof of Theorem 2. □

The result of Theorem 5 can be partially extended to certain variable co-efficient elliptic problems but not to the problem of Theorem 2 if the matrix M has variable coefficients. In the general case there is no guarantee that the required system of first-order partial differential equations can be solved. Of course these solutions have been used to construct the operators B_j. However we are able to extend the idea to certain diagonal elliptic problems.

Theorem 6. *Let* $\Omega \subseteq \mathbb{R}^m$ *be a bounded region in which the problem*

(3.49) $$Au = -\sum_{j=1}^{m} \frac{\partial}{\partial x_j}\left(d_j^2 \frac{\partial u}{\partial x_j}\right) + gu = \lambda u \quad in\ \Omega; \qquad u = 0 \quad on\ \partial\Omega$$

has discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots$ *. In* (3.49) *the coefficient functions* d_j *are assumed to be positive and differentiable functions of* x_j *only and* g *is any function on* Ω *. If* M *is a real constant so that*

$$g(x) - \frac{1}{4}\sum_{j=1}^{m}\left(\left(\frac{\partial d_j}{\partial x_j}\right)^2 - \frac{\partial^2}{\partial x_j^2}(d_j^2)\right) \geq M$$

in Ω, *then the eigenvalues satisfy*

(3.50) $$\sum_{i=1}^{n} \frac{\lambda_i - M}{\lambda_{n+1} - \lambda_i} \geq \frac{mn}{4}.$$

Proof. In Theorem 1 we take

(3.51a) $$T_j u = d_j \frac{\partial u}{\partial x_j} + \frac{1}{2}\frac{\partial d_j}{\partial x_j}u \quad for\ j = 1, 2, \ldots, m$$

and

$$(3.51b) \qquad T_{m+1}u = \left(i\sqrt{g - \frac{1}{4}\sum_{j=1}^{m}\left(\left(\frac{\partial d_j}{\partial x_j}\right)^2 - \frac{\partial^2}{\partial x_j^2}(d_j^2)\right)} - M \right)u.$$

It is easy to show that the operator A is symmetric in the inner product of (3.29) and the operators of (3.51) are skew-symmetric. We choose the operators B_j by

$$(3.52a) \qquad B_j u = \phi_j u \quad \text{for } j = 1, 2, \ldots, m,$$

where $d_j \partial \phi_j / \partial x_j = -1$ in Ω, $\partial \phi_j / \partial x_k = 0$ if $k \neq j$ and

$$(3.52b) \qquad B_{m+1}u = -u.$$

Such ϕ_j exist since the d_j depend only on x_j. The operators B_j of (3.52) are seen to be symmetric in the inner product (3.29). We also have

$$(3.53a) \qquad \langle T_j u, T_j u \rangle = -\int_{\Omega} \frac{\partial}{\partial x_j} d_j^2 \frac{\partial u}{\partial x_j}\overline{u} - \frac{1}{4}\int_{\Omega}\left(\frac{\partial^2(d_j^2)}{\partial x_j^2} - \left(\frac{\partial d_j}{\partial x_j}\right)^2\right)|u|^2$$

for $j = 1, 2, \ldots, m$ and

$$(3.53b)$$
$$\langle T_{m+1}u, T_{m+1}u \rangle = \int_{\Omega}\left(g - \frac{1}{4}\sum_{j=1}^{m}\left(\left(\frac{\partial d_j}{\partial x_j}\right)^2 - \frac{\partial^2}{\partial x_j^2}(d_j)^2\right) - M\right)|u|^2.$$

Hence

$$(3.53c) \qquad \sum_{j=1}^{m+1}\langle T_j u, T_j u \rangle = \langle Au, u \rangle - M\langle u, u \rangle.$$

Looking at the B_j we have

$$(3.54a) \qquad \langle [T_j, B_j]u, u \rangle = -\langle u, u \rangle \quad \text{for } j = 1, 2, \ldots, m$$

and

$$(3.54b) \qquad \langle [T_{m+1}, B_{m+1}]u, u \rangle = 0.$$

Finally we have

$$(3.55) \qquad \sum_{j=1}^{m+1}\langle [[A, B_j]B_j]u, u \rangle = -2m\langle u, u \rangle.$$

With the results of (3.53)–(3.55), equation (2.1) becomes

$$(3.56) \qquad \gamma(-nm) + \gamma^2 \sum_{i=1}^{n}\frac{\lambda_i - M}{\beta - \lambda_i} = -mn.$$

Equation (3.56) is in the same form as (3.18) and the result follows as before.   □

It is not clear whether a similar result may be obtained directly from (2.1) if any of the functions d_j are allowed to depend on any coordinate other than x_j.

The results of this section are all implicit bounds on the eigenvalues of the various problems. Each give rise to explicit bounds. Inequalities (3.17) and (3.35) each imply that

$$(3.57) \qquad \lambda_{n+1} \leq \lambda_n + \frac{4}{mn} \sum_{i=1}^{n} \lambda_i$$

since the sum in each is bounded above by

$$(3.58) \qquad \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_n} \geq \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \geq \frac{mn}{4}.$$

Factoring the constant $1/(\lambda_{n+1} - \lambda_n)$ out of the sum in (3.58) and rearranging yields the desired result. Similarly inequality (3.30) implies that

$$(3.59) \qquad \lambda_{n+1} < \lambda_n + \frac{4}{mn} \sum_{i=1}^{n} \lambda_i - \frac{4\|\mathbf{w}\|^2}{m}.$$

The other results of this section yield similar types of explicit inequalities.

Notice that each problem considered in this section has the same boundary condition: $u = 0$ on $\partial\Omega$. This was necessary since the B_j always turned out to be multiplication operators and this boundary condition is the only one preserved under multiplication by arbitrary functions. The fact that the T_j are first-order differential operators for the most part and the B_j multiplication operators seems to be basic and this author has not yet found any situation where Theorem 1 provides a useful application where this is not the case. In the following section we consider some higher-order problems.

4. HIGHER-ORDER PROBLEMS

So far we have only seen applications in which the sum $\sum_{j=1}^{M} \langle T_j u, T_j u \rangle = \langle Au, u \rangle + \text{constant}\langle u, u \rangle$. For higher-order problems this does not turn out to be the case. The results in this section depend on a theorem of this author [5] as follows.

Proposition 2. *Let V be a real or complex inner product space with inner product $(\cdot, \cdot)$. Let D be a linear submanifold of V and let $Q: D \to V$ be a linear operator in V. Let n be a positive integer and let u be a fixed vector in the domain of Q^n. Suppose that for all integers r and p which satisfy $0 \leq r \leq p \leq n$, the equality $|(Q^p u, u)| = |(Q^{p-r} u, Q^r u)|$ holds. Then* (i) *for each integer k and even integer m which satisfy $0 \leq k \leq m \leq n$, the inequality*

$$(4.1) \qquad |(Q^k u, u)| \leq |(Q^m u, u)|^{k/m} (u, u)^{(m-k)/m}$$

holds.

(ii) *Inequality* (4.1) *is satisfied for each integer* k *and odd integer* m *which satisfy* $0 \le k \le m \le n$ *if in addition to the above hypotheses there is a finite collection of operators* $\{T_j : j = 1, 2, \ldots, M\}$ *in* V *which satisfy the following: For each pair of integers* r *and* p *so that* $1 \le r \le p \le n$ *the equality*

$$|(Q^p u, u)| = \left| \sum_{j=1}^{M} (T_j Q^{p-r} u, T_j Q^{r-1} u) \right|$$

holds.

Proof. See [5] or [4].

The result of Proposition 2 can be used in conjunction wth Theorem 1 after observing the following.

Proposition 3. *Suppose that* M *is a positive integer and that* Q, T_j, *and* B_j, $j = 1, 2, \ldots, M$, *are linear operators in an inner product space. If the commutators* $[Q, T_j] = 0$ *and* $[T_j, B_i] = -\delta_{ij}$ *for* $i, j = 1, 2, \ldots, M$ *and if* $Q = -\sum_{j=1}^{M} T_j^2$ *then for all positive integers* n,

$$(4.2) \qquad \sum_{j=1}^{M} [[Q^n, B_j], B_j] = -2n(M + 2(n-1))Q^{n-1}.$$

Proof. First we claim that

$$(4.3) \qquad [Q^n, B_j] = 2nQ^{n-1}T_j$$

for all $j = 1, 2, \ldots, M$ and all positive n. We proceed by induction on n for a fixed j. We see that $[Q^1 B_j] = 2Q^0 T_j = 2T_j$ in the proof of Proposition 1. For the inductive step notice that if (4.3) holds for some n then

$$(4.4) \qquad [Q^{n+1}, B_j] = Q[Q^n, B_j] + [Q, B_j]Q^n$$
$$= 2QnQ^{n-1}T_j + 2T_j Q^n = 2(n+1)Q^n T_j$$

as required. By (4.3) we have

$$(4.5) \; \sum_{j=1}^{M}[[Q^n, B_j], B_j] = \sum_{j=1}^{M}[2nQ^{n-1}T_j, B_j]$$

$$= 2nQ^{n-1}\sum_{j=1}^{M}[T_j, B_j] + 2n\sum_{j=1}^{M}[Q^{n-1}, B_j]T_j$$

$$= -2nMQ^{n-1} + 2n\sum_{j=1}^{M}(2(n-1)Q^{n-2}T_j T_j)$$

$$= -2nMQ^{n-1} - 4n(n-1)Q^{n-1} = -2n(M + 2(n-1))Q^{n-1}. \quad \square$$

Proposition 3 allows us to take the operator A in Theorem 1 to be Q^n and if we can write $Q = -\sum_{j=1}^{M} T_j^2$ where the T_j are skew-symmetric, we can

use Proposition 2 to obtain a suitable estimate for the right-hand side of (2.1). This enables us to get results similar to those of §3 for higher-order problems. Our principal application is to polynomials in the Laplacian. We restrict our consideration to the Laplacian for simplicity even though the Laplacian can be replaced with the operator of (3.1). Several earlier results by Hile and Yeh [3] and Chen [1] were established for certain even degree monomials in the Laplacian of the form

$$(4.6) \qquad \Delta^{2l} u = \lambda u \quad \text{in } \Omega \subseteq \mathbb{R}^m, \qquad u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{2l-1} u}{\partial n^{2l-1}} = 0 \quad \text{on } \partial\Omega,$$

where l is one of a class of positive integers. This author [4] later improved and extended the earlier results of problem (4.6) to arbitrary polynomials in the Laplacian using Proposition 2.

Theorem 7. *Let $p(x) = \sum_{i=0}^{l} a_i x^i$ be a monic polynomial and $q(x) = \sum_{i=0}^{r} b_i x^i$ be a polynomial. Suppose that $a_i \geq 0$ and $b_i \geq 0$ for all i and $r < l$. Suppose that $\Omega \subseteq \mathbb{R}^m$ is a region in which the problem*

$$(4.7a) \qquad\qquad (p(-\Delta) - q(-\Delta))u = \lambda u \quad \text{in } \Omega,$$

$$(4.7b) \qquad\qquad u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{l-1} u}{\partial n^{l-1}} = 0 \quad \text{on } \partial\Omega$$

is selfadjoint with discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots$. Then the eigenvalues of (4.7) satisfy

$$(4.8) \qquad \left(\sum_{i=1}^{n} \frac{\alpha_{i,1}}{\lambda_{n+1} - \lambda_i} \right) \left(\sum_{i=1}^{n} \sum_{k=1}^{l} (m + 2k - 2) k a_k \alpha_{i,k-1}^{k-1} \right) > \frac{m^2 n^2}{4}.$$

In (4.8) we define $\alpha_{i,0} = 1$ and for $j \geq 1$, $\alpha_{i,j}$ is the largest (positive) root of

$$(4.9a) \qquad a_0 + \sum_{k=j}^{l} a_k x^k - q(x) - \lambda_i = 0 \quad \text{if } r \leq j \leq l,$$

$$(4.9b) \qquad a_0 + \sum_{k=j+1}^{r} b_k \alpha_{i,k}^k - \sum_{k=j}^{l} a_k x^k - \sum_{k=0}^{j} b_k x^k - \lambda_i = 0 \quad \text{if } 1 \leq j \leq r - 1.$$

Proof. Here we choose the operator Q of Propositions 2 and 3 by $Q = -\Delta$, $T_j u = \partial u / \partial x_j$ and $B_j u = -x_j u$. Finally we choose the operator A of Theorem 1 to be $A = p(-\Delta) - q(-\Delta) = p(Q) - q(Q)$. These operators satisfy all of the hypotheses of Theorem 1 and Propositions 2 and 3 so (2.1) becomes
(4.10)

$$-\gamma nm + \gamma^2 \sum_{i=1}^{n} \frac{\langle Q u_i, u_i \rangle}{\beta - \lambda_i} = -\frac{1}{2} \sum_{i=1}^{n} \langle (2m(p-q)'(Q) + 4Q(p-q)''(Q)) u_i, u_i \rangle.$$

First we want to find a lower bound for the right-hand side of (4.10). Proposition 2 shows that $\langle Q^k u_i, u_i \rangle \leq \langle Q^j u_i, u_i \rangle^{k/j}$ if $0 \leq k \leq j \leq l$. Thus if we fix

i and let $\beta_k^k = \langle Q^k u_i, u_i \rangle$ for $k = 1, 2, \ldots, l$, Proposition 2 implies that $0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_l$, where β_j is the principal root of β_j^j. (We take $\beta_0 = 1 = \langle Q^0 u_i, u_i \rangle$.)

For $j \ge r$ we have

$$(4.11) \qquad \lambda_i = \sum_{k=0}^{l} a_k \beta_k^k - \sum_{k=0}^{r} b_k \beta_k^k \ge \sum_{k=j}^{l} a_k \beta_k^k + a_0 - \sum_{k=0}^{r} b_k \beta_k^k$$

$$\ge \sum_{k=j}^{l} a_k \beta_j^k + a_0 - \sum_{k=0}^{r} b_k \beta_j^k = \sum_{k=j}^{l} a_k \beta_j^k + a_0 - q(\beta_j).$$

Now consider the functions $f_j(x) = a_0 + \sum_{k=j}^{l} a_k x^k - q(x)$ for $r \le j \le l$. Now $f_j(x) \to +\infty$ as $x \to +\infty$ and is continuous and (4.11) asserts that $f_j(\beta_j) \le \lambda_i$. The continuity of f and limiting behavior imply the existence of a number which we have called $\alpha_{i,j}$ so that $f_j(\alpha_{i,j}) = \lambda_i$ and for this root we have $\alpha_{i,j} \ge \beta_j$.

For $1 \le j \le r - 1$ we have

$$(4.12) \qquad \lambda_i = \sum_{k=0}^{l} a_k \beta_k^k - \sum_{k=0}^{r} b_k \beta_k^k$$

$$= \sum_{k=0}^{l} a_k \beta_k^k - \sum_{k=0}^{j} b_k \beta_k^k - \sum_{k=j+1}^{r} b_k \beta_k^k$$

$$\ge \sum_{k=j}^{l} a_k \beta_j^k + a_0 - \sum_{k=0}^{j} b_k \beta_j^k - \sum_{k=j+1}^{r} b_k \alpha_{i,k}^k,$$

where the $\alpha_{i,k}$ are those found from (4.9b). The argument that (4.12) leads to (4.9b) follows the argument for (4.9a) above.

Now

(4.13)

$$\langle (m(p-q)'(Q) + 2Q(p-q)''(Q))u_i, u_i \rangle \le \langle (mp'(Q) + 2Qp''(Q))u_i, u_i \rangle$$

$$= m \sum_{k=1}^{l} k a_k \beta_{k-1}^{k-1} + 2 \sum_{k=1}^{l} k(k-1) a_k \beta_{k-1}^{k-1}$$

$$\le \sum_{k=1}^{l} k(m + 2k - 2) a_k \alpha_{i,k-1}^{k-1}.$$

Inequality (4.13) yields the desired lower bound for the right-hand side of (4.10) and $\alpha_{i,1} \ge \langle Qu_i, u_i \rangle$. Thus (4.10) can be rewritten

$$(4.14) \qquad -\gamma mn + \gamma^2 \sum_{i=1}^{n} \frac{\alpha_{i,1}}{\beta - \lambda_i} \ge -\sum_{i=1}^{n} \sum_{k=1}^{l} k(m + 2k - 2) a_k \alpha_{i,k-1}^{k-1}.$$

In (4.14), β is the same number which yields equality in (4.10). Evidently equality is attained in (4.14) for a larger value of β. In particular if $\gamma > 0$ and

$\beta > \lambda_n$ yield equality in (4.14) we must have $\beta \geq \lambda_{n+1}$. Again, minimizing the left-hand side of (4.14) in γ as in Proposition 2 and rearranging yields the desired result (4.8). As before, the strict inequality comes from the fact that for each j, $\partial u_i / \partial x_j$ will not satisfy the boundary conditions of (4.7b) if u_i is an eigenvector of (4.7a). □

The result (4.8) is at best difficult to understand along with the computation of the $\alpha_{i,k}$. A more explicit form of (4.8) is available if we take $q = 0$.

Theorem 8. *Suppose the hypotheses of Theorem 7 except that $q = 0$ and $a_0 = 0$. Then the eigenvalues satisfy*

$$(4.15) \qquad \left(\sum_{i=1}^{n} \frac{\lambda_i^{1/l}}{\lambda_{n+1} - \lambda_i} \right) \left(\sum_{i=1}^{n} \frac{m}{2} p'(\lambda_i^{1/l}) + \lambda_i^{1/l} p''(\lambda_i^{1/l}) \right) > \frac{m^2 n^2}{8}.$$

Proof. In the notation of the proof of Theorem 7 we see that when $j = l$, (4.11) reduces to

$$(4.16) \qquad \lambda_i \geq \beta_l^l.$$

Thus $\beta_l \leq \lambda_i^{1/l}$. Also for $0 \leq j \leq l$ we have $\beta_j^j \leq \beta_l^j$. (The extension to $j = 0$ is easy since $1 \leq 1$.) Finally $\beta_l \leq \alpha_{i,l} \leq \lambda_i^{1/l}$. Writing out the right-hand side of (4.14) with $\alpha_{i,j} = \alpha_{i,l}$ for all j, $0 \leq j \leq l$, expressing it in terms of derivatives of p and following the remainder of the proof of Theorem 7 yields the result. □

At this point we look at a corollary of Theorem 8 and show that the earlier results of Hile and Yeh [3] and Chen [1] follow from this corollary.

Corollary 3. *In Theorem 8 if $p(x) = x^l$, $l > 0$ then the inequality*

$$(4.17) \qquad \left(\sum_{i=1}^{n} \frac{\lambda_i^{1/l}}{\lambda_{n+1} - \lambda_i} \right) \left(\sum_{i=1}^{n} \lambda_i^{(l-1)/l} \right) > \frac{m^2 n^2}{4l(m + 2l - 2)}$$

is satisfied.

Proof. Comparison of (4.17) and (4.15) shows that we need only show that for $p(x) = x^l$ we have

$$(4.18) \qquad \frac{m}{2} p'(\lambda_i^{1/l}) + \lambda_i^{1/l} p''(\lambda_i^{1/l}) = \frac{1}{2} \lambda_i^{(l-1)/l} l(m + 2l - 2).$$

This is an easy task. □

The results of [3] and [1] are as follows.

Proposition 4. *For certain even integers l (the condition is irrelevant here) the eigenvalues of the problem in Corollary 3 satisfy*

$$(4.19) \qquad \left(\sum_{i=1}^{n} \frac{\lambda_i^{1/l}}{\lambda_{n+1} - \lambda_i} \right) \left(\sum_{i=1}^{n} \lambda_i \right)^{(l-1)/l} n^{1/l} \geq \frac{m^2 n^2}{4l(m + 2l - 2)}.$$

The result (4.19) follows easily from (4.17) by way of the Hölder inequality which for our purposes reads:

$$(4.20) \qquad \sum_{i=1}^{n} \lambda_i^{(l-1)/l} \leq n^{1/l} \left(\sum_{i=1}^{n} \lambda_i \right)^{(l-1)/l} .$$

The theorems and corollaries above also hold if the Laplacian, at least $-\Delta$, is replaced with the operator of Theorem 2. Furthermore, similar results can be obtained for polynomials in the operator of Theorem 3. The problem of Theorem 4 presents an interesting possibility.

Theorem 9. *Let $p(x)$ be a monic polynomial of degree $l > 0$ with nonnegative coefficients and zero constant term, $\Omega \subseteq \mathbb{R}^m$ and ρ any function and M a constant with $\rho \geq M$ on Ω. If Ω is such that the problem*

$$(4.21) \quad p(-\Delta)u + \rho(x)u = \lambda u \quad in \; \Omega; \qquad u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{l-1} u}{\partial n^{l-1}} = 0 \quad on \; \partial\Omega$$

is selfadjoint with discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots$ then the eigenvalues of (4.21) satisfy
(4.22)

$$\left(\sum_{i=1}^{n} \frac{(\lambda_i - M)^{1/l}}{\lambda_{n+1} - \lambda_i} \right) \left(\sum_{i=1}^{n} mp'((\lambda_i - M)^{1/l}) + 2(\lambda_i - M)^{1/l} p''((\lambda_i - M)^{1/l}) \right)$$

$$> \frac{m^2 n^2}{4}$$

if $\lambda_1 - M \geq 0$.

Proof. Here in Propositions 2 and 3 we take T_j and B_j as in Theorem 4 except that $B_{m+1} = T_{m+1} = 0$, $Q = -\Delta$ and in Theorem 1, $Au = (p(-\Delta) + \rho(x))u$. Proposition 3 is not directly applicable in this case. However as before we have

$$(4.23) \qquad [[A, B_j], B_j] = [[p(-\Delta), B_j], B_j] \quad for \; j = 1, 2, \ldots, m+1.$$

Since the B_j and $Q = -\Delta$ satisfy the hypotheses of Proposition 3 we get

$$(4.24) \qquad \sum_{j=1}^{m} \langle [[A, B_j], B_j] u_i, u_i \rangle = \sum_{j=1}^{m} \langle [[p(-\Delta), B_j], B_j] u_i, u_i \rangle$$

$$= -2 \langle (mp'(-\Delta) + 2(-\Delta)p''(-\Delta)) u_i, u_i \rangle .$$

Also

$$(4.25) \qquad \sum_{j=1}^{m} \langle [T_j, B_j] u_i, u_i \rangle = -m \langle u_i, u_i \rangle$$

and

$$(4.26) \qquad \sum_{j=1}^{m} \langle T_j u_i, T_j u_i \rangle = \langle (-\Delta) u_i, u_i \rangle .$$

Putting the results of (4.24)–(4.26) into (2.1) leaves us with

$$(4.27) \quad \gamma(-nm) + \gamma^2 \sum_{i=1}^{n} \frac{\langle (-\Delta)u_i, u_i \rangle}{\beta - \lambda_i} = -\langle (mp'(-\Delta) + 2(-\Delta)p''(-\Delta))u_i, u_i \rangle.$$

By the proof of Theorem 7 for each i we have

$$\langle (-\Delta)^k u_i, u_i \rangle^{1/k} \leq \langle (-\Delta)^l u_i, u_i \rangle \leq \langle p(-\Delta)u_i, u_i \rangle$$
$$= \langle (\lambda_i - p)u_i, u_i \rangle \leq \langle (\lambda_i - M)u_i, u_i \rangle = \lambda_i - M$$

for $1 \leq k \leq l$. Thus

$$(4.29a) \qquad\qquad \langle -\Delta u_i, u_i \rangle \leq (\lambda_i - M)^{1/l}$$

and

$$(4.29b) \quad \langle (mp'(-\Delta) + 2(-\Delta)p''(-\Delta))u_i, u_i \rangle$$
$$\leq mp'((\lambda_i - M)^{1/l}) + 2(\lambda_i - M)^{1/l}p''((\lambda_i - M)^{1/l}).$$

Placing the estimates of (4.29) into (4.27), using the same argument as was used in the proof of Theorem 7 yields the result. □

Remarks. (i) In Theorem 9 if $M = \inf\{p(x): x \in \Omega\}$, the quantities $\lambda_1 - M \geq 0$ in Ω is always satisfied.

(ii) A sharper estimate can be attained in (4.28) as is done in the proof of Theorem 7, but we have chosen the simpler argument for the sake of simplicity.

The operators of Theorems 2 and 3 can be combined into higher-order problems through a polynomial as above as well as the operators of Theorems 5 and 6 but the number of combinations is so large that it is hopeless to list all the possibilities as well as the variety of estimates available. However a problem for which Theorem 1 is not directly applicable but which follows as a corollary to our earlier problems is

Corollary 4. *Let $p(x)$ be a polynomial with nonnegative coefficients of degree $l > 0$ and $\Omega \subseteq \mathbb{R}^m$ a bounded domain. If Ω is such that*

$$(4.30) \qquad\qquad p(-\Delta)u = \lambda u \quad in \ \Omega;$$

$$(4.31) \qquad u = \Delta u = \Delta^2 u = \cdots = \Delta^{l-1} u = 0 \quad on \ \partial\Omega$$

is selfadjoint with discrete spectrum then

$$(4.32) \qquad\qquad \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_{n+1} - \alpha_i} > \frac{mn}{4},$$

where α_i is the (only) positive number so that $p(\alpha_i) = \lambda_i$.

Proof. The eigenvalues, λ_i of (4.30), (4.31) satisfy

$$(4.33) \qquad\qquad p(\alpha_i) = \lambda_i,$$

where α_i is the ith eigenvalue of the problem of (3.26). The result (4.32) then follows easily from Corollary 1. □

A variety of other boundary conditions for problem (4.30) simply yield problems whose eigenvalues are some function of the eigenvalues of one of the earlier problems in either of §§3 or 4. Each such problem yields a result similar to that of Corollary 4. As a final note for higher-order problems, explicit bounds can again be established from the implicit bounds given here. For example, (4.15) yields, by the same argument as was used to establish (3.57) and (3.59),

$$(4.34) \qquad \lambda_{n+1} < \lambda_n + \frac{8}{m^2 n^2} \left(\sum_{i=1}^{n} \lambda_i^{1/l} \right) \left(\sum_{i=1}^{n} \frac{m}{2} p'(\lambda_i^{1/l}) + \lambda_i^{1/l} p''(\lambda_i^{1/l}) \right).$$

So we see that for the type of higher-order problem considered here the rate of growth in the eigenvalues is bounded above in a way which has no explicit dependence on the domain Ω.

5. A SYSTEM OF EQUATIONS

Our last task is to show how to apply Theorem 1 to a system. Let α be a constant, $\alpha \geq 0$. We consider the eigenvalue problem

$$(5.1) \qquad A\mathbf{u} = -\Delta\mathbf{u} - \alpha \operatorname{grad}(\operatorname{div}(\mathbf{u})) = \Lambda\mathbf{u} \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega$$

which governs the behavior of an elastic medium. We begin with a few preliminaries.

Lemma 3. *In the inner product* $\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \mathbf{u} \cdot \mathbf{v}$, *which we call* $L^2(\Omega)$, *define the operators*

$$(5.2) \qquad R_j\mathbf{u} = \operatorname{div}(\mathbf{u}) \operatorname{grad} x_j + \operatorname{grad} u_j \quad \text{for } j = 1, 2, \ldots, m$$

and

$$(5.3) \qquad S_j\mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_j} \quad \text{for } j = 1, 2, \ldots, m,$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ *and* $\mathbf{u} = \mathbf{0}$ *on* $\partial\Omega$. *The symbol* x_j *is the projection of a vector* $\mathbf{x}$ *onto its* jth *component. The operators* R_j *and* S_j *satisfy*

$$(5.4a) \qquad \sum_{j=1}^{m} \langle R_j\mathbf{u}, R_j\mathbf{u} \rangle = -\langle (m+2) \operatorname{grad} \operatorname{div} \mathbf{u} + \Delta\mathbf{u}, \mathbf{u} \rangle$$

and

$$(5.4b) \qquad \sum_{j=1}^{m} \langle S_j\mathbf{u}, S_j\mathbf{u} \rangle = -\langle \Delta\mathbf{u}, \mathbf{u} \rangle.$$

Proof. If $\mathbb{R}^m$ we have

$$(5.5) \qquad \operatorname{div}(g\mathbf{u}) = \operatorname{grad} g \cdot \mathbf{u} + g \operatorname{div} \mathbf{u}$$

and the divergence theorem

$$(5.6) \qquad \int_\Omega \operatorname{div} \mathbf{u} = \int_{\partial\Omega} \mathbf{u} \cdot dS.$$

If $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ then (5.5) and (5.6) may be combined to yield

(5.7) $$\int_\Omega \operatorname{grad} g \cdot \mathbf{u} = -\int_\Omega g \operatorname{div} \mathbf{u}.$$

The definition of R_j shows that

(5.8) $$\langle R_j \mathbf{u}, R_j \mathbf{u} \rangle = \int_\Omega \| \operatorname{div} \mathbf{u} \operatorname{grad} x_j + \operatorname{grad} u_j \|^2$$
$$= \int_\Omega (\operatorname{div} \mathbf{u})^2 + 2 \int_\Omega \operatorname{div} \mathbf{u} \frac{\partial u_j}{\partial x_j} + \int_\Omega \| \operatorname{grad} u_j \|^2.$$

Application of (5.7) with the identification $\mathbf{u} \sim \operatorname{grad} u_j$ and $g \sim u_j$ we see that

$$\int_\Omega \| \operatorname{grad} u_j \|^2 = -\int_\Omega (\operatorname{div} \operatorname{grad} u_j) u_j = -\int_\Omega u_j \Delta u_j.$$

Thus we see that

(5.9) $$\sum_{j=1}^m \langle R_j \mathbf{u}, R_j \mathbf{u} \rangle = m \int_\Omega (\operatorname{div} \mathbf{u})^2 + 2 \int_\Omega (\operatorname{div} \mathbf{u})^2 - \int_\Omega \mathbf{u} \cdot \Delta \mathbf{u}.$$

Finally, applicaton of (5.7) with $g \sim \operatorname{div} \mathbf{u}$, (5.9) becomes

(5.10) $$\sum_{j=1}^m \langle R_j \mathbf{u}, R_j \mathbf{u} \rangle = -\int_\Omega \mathbf{u} \cdot ((m+2) \operatorname{grad} \operatorname{div}(\mathbf{u}) + \Delta \mathbf{u}).$$

Equation (5.10) is equivalent to (5.4a).

For the S_j we have

(5.11) $$\langle S_j \mathbf{u}, S_j \mathbf{u} \rangle = \int_\Omega \| \frac{\partial \mathbf{u}}{\partial x_j} \|^2 = -\int_\Omega \mathbf{u} \cdot \frac{\partial^2 \mathbf{u}}{\partial x_j^2}.$$

Summing (5.11) over j and using the definition of Δ yields the result (5.4b). $\square$

Lemma 4. *Consider the subset* $\{\mathbf{u} \in L^2(\Omega) : \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\} = D$. *Then the operators*

(5.12) $$C_j \mathbf{u} = -x_j \mathbf{u} \quad \text{for } j = 1, 2, \dots, m$$

are symmetric on $L^2(\Omega)$ *and leave* D *invariant. Moreover the commutators with the* R_j *and* S_j *of Lemma 3 obey*

(5.13a) $$\sum_{j=1}^m [R_j, C_j] \mathbf{u} = -2\mathbf{u}$$

and

(5.13b) $$\sum_{j=1}^m [S_j, C_j] \mathbf{u} = -m\mathbf{u}.$$

Proof. We have $[R_j, C_j]\mathbf{u} = R_j C_j \mathbf{u} - C_j R_j \mathbf{u}$. Thus

(5.14)
$$[R_j, C_j]\mathbf{u} = \operatorname{div}(-x_j \mathbf{u}) \operatorname{grad} x_j + \operatorname{grad}(-x_j u_j) + x_j \operatorname{div} \mathbf{u} \operatorname{grad} x_j + x_j \operatorname{grad} u_j$$
$$= -u_j \operatorname{grad} x_j - (\operatorname{grad} x_j) u_j = -2u_j \operatorname{grad} x_j.$$

By summing (5.14) over j we obtain (5.13a).

For the S_j we have

(5.15)
$$[S_j, C_j]\mathbf{u} = \frac{\partial}{\partial x_j}(-x_j \mathbf{u}) + x_j \frac{\partial u}{\partial x_j} = -\mathbf{u}.$$

Summing (5.15) over j yields the result (5.13b). The fact that the C_j are symmetric on $L^2(\Omega)$ and leave D invariant is easily seen. □

Lemma 5. *For the operators* $L\mathbf{u} = -\Delta \mathbf{u}$ *and* $M\mathbf{u} = -\operatorname{grad} \operatorname{div} \mathbf{u}$ *and the operators* C_j *of Lemma* 4 *we have*

(5.16a)
$$[L, C_j]\mathbf{u} = 2S_j \mathbf{u}$$

and

(5.16b)
$$[M, C_j]\mathbf{u} = R_j \mathbf{u},$$

where the operators S_j *and* R_j *are as in Lemma* 3. *Moreover the* R_j *and* S_j *are skew-symmetric.*

Proof. The ith component of $[L, C_j]\mathbf{u}$ is given by

(5.17)
$$-\Delta(-x_j u_i) - (-x_j(-\Delta u_i)) = \Delta(x_j u_i) - x_j \Delta u_i$$
$$= 2 \operatorname{grad} x_j \cdot \operatorname{grad} u_i + x_j \Delta u_i - x_j \Delta u_i = 2\partial u_i/\partial x_j.$$

Since (5.17) holds for each component of $[L, C_j]\mathbf{u}$ we see that $[L, C_j]\mathbf{u} = 2\partial \mathbf{u}/\partial x_j = 2S_j \mathbf{u}$. We use (5.5) to establish (5.16b). We have

(5.18) $[M, C_j]\mathbf{u} = -\operatorname{grad} \operatorname{div}(-x_j \mathbf{u}) - x_j \operatorname{grad} \operatorname{div} \mathbf{u}$
$$= \operatorname{grad}(\operatorname{grad} x_j \cdot \mathbf{u}) + \operatorname{grad}(x_j \operatorname{div} \mathbf{u}) - x_j \operatorname{grad} \operatorname{div} \mathbf{u}$$
$$= \operatorname{grad} u_j + \operatorname{div} \mathbf{u} \operatorname{grad} x_j + x_j \operatorname{grad} \operatorname{div} \mathbf{u} - x_j \operatorname{grad} \operatorname{div} \mathbf{u}$$
$$= \operatorname{div} \mathbf{u} \operatorname{grad} x_j + \operatorname{grad} u_j = R_j \mathbf{u}.$$

The R_j and S_j are skew-symmetric because they are the commutators of symmetric operators. □

Theorem 10. *If* $\Omega \subseteq \mathbb{R}^m$ *is bounded,* $\alpha \geq 0$ *is a constant and problem* (5.1) *is selfadjoint in* Ω *with* $\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_n < \Lambda_{n+1} \leq \cdots$ *then the eigenvalues of* (5.1) *satisfy*

(5.19)
$$\sum_{i=1}^{n} \frac{\Lambda_i}{\Lambda_{n+1} - \Lambda_i} \geq \frac{m^2 n}{4(m + \alpha)}.$$

Proof. We have established all that we need in Lemmas 3–5 above. All we need to do is to choose the B_j, T_j, and γ in (2.1) properly. Define

(5.20) $$T_j = S_j \quad \text{for } j = 1, 2, \ldots, m.$$

Thus Lemma 3 implies that

(5.21) $$\sum_{j=1}^{m} \langle T_j \mathbf{u}, T_j \mathbf{u} \rangle = \langle -\Delta \mathbf{u}, \mathbf{u} \rangle \leq \langle A \mathbf{u}, \mathbf{u} \rangle.$$

Next we define the B_j by

(5.22) $$B_j = C_j \quad \text{for } j = 1, 2, \ldots, m.$$

Thus using Lemma 4, (5.20), and (5.22) we obtain

(5.23) $$\sum_{j=1}^{m} \langle [T_j, B_j] \mathbf{u}, \mathbf{u} \rangle = -m \langle \mathbf{u}, \mathbf{u} \rangle.$$

Since $[A, C_j] = [L, C_j] + \alpha[M, C_j]$, where L and M are defined in Lemma 5, we have $[A, C_j] = 2S_j + \alpha R_j$. Thus, since $B_j = C_j$

(5.24) $$\sum_{j=1}^{m} [[A, B_j], B_j] \mathbf{u} = 2 \sum_{j=1}^{m} [S_j, B_j] \mathbf{u} + \alpha \sum_{j=1}^{m} [R_j, B_j] \mathbf{u}$$
$$= -2(m + \alpha) \mathbf{u}.$$

Placing $\mathbf{u}_i$, a normalized eigenvector of A corresponding to Λ_i, into (5.21), (5.23), and (5.24) and then the results into (2.1) we obtain

(5.25) $$-mn\gamma + \gamma^2 \sum_{i=1}^{n} \frac{\langle -\Delta \mathbf{u}_i, \mathbf{u}_i \rangle}{\beta - \Lambda_i} = -(m + \alpha)n.$$

If β solves (5.25) then

(5.26) $$-mn\gamma + \gamma^2 \sum_{i=1}^{n} \frac{\Lambda_i}{\beta - \Lambda_i} \geq -n(m + \alpha)$$

by the upper bound in (5.21). Finally if we find a value for β yielding equality in (5.26) it must be no less than the value of β which solves (5.25).

Minimizing the left-hand side of (5.26) in γ as before, and rearranging yields

(5.27) $$\sum_{i=1}^{n} \frac{\Lambda_i}{\beta - \Lambda_i} = \frac{m^2 n}{4(m + \alpha)}.$$

Equation (5.27) has a solution $\beta > \Lambda_n$ so we can conclude that

(5.28) $$\sum_{i=1}^{n} \frac{\Lambda_i}{\Lambda_{n+1} - \Lambda_i} \geq \frac{m^2 n}{4(m + \alpha)},$$

the desired result. $\square$

An earlier result of this author [6] for problem (5.1) was obtained in explicit form

$$(5.29) \qquad \Lambda_k \leq \left(\frac{m + \alpha k}{m}\right) \left(\Lambda_{(j-1)m+1} + \frac{4}{mj} \sum_{i=1}^{j} \Lambda_{(i-1)m+1}\right)$$

where j is the largest integer which does not exceed k/m. It is difficult to compare result (5.19) to (5.29) in general. However for $k = m = 2$, (5.29) becomes

$$(5.30) \qquad \Lambda_2 \leq (1 + \alpha)3\Lambda_1,$$

and (5.19) can be rearranged to become

$$(5.31) \qquad \Lambda_2 \leq \left(1 + \frac{4(2 + \alpha)}{4}\right)\Lambda_1 = (3 + \alpha)\Lambda_1.$$

Clearly the upper bound in (5.31) is smaller than that of (5.30) in this case.

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